Queueing Theory Modelling and Applications

Abstract

In this paper we study the building blocks and discuss M/ M/1/ N queueing model and its applications. We discuss some steady-state parameters of M/M/1/N model and matrix geometric method.. Some queues are analyzed in terms of steady-state derivation and Matrix geometric method.

Keywords: Queueing system, building blocks, applications, steady-state, matrix geometric.

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Introduction

Each of us had spent a great deal of time waiting in lines. In this paper we will take a brief look into the formation of queueing theory along with some examples of queueing models and their applications. The queueing behaviour can be analyzed mathematically by the performance measurement calculation. These measures are very important because they are often relevant with the work efficiency or economic losses of real queueing system. In the queueing system, analysis is more accurate and deterministic results are expected.

The first text book on the subject: Queues, Inventories and Maintenance was written in [1958] by Morse,P. Satty, T.L., wrote his famous book: Elements of Queueing Theory with Applications in [1961]. Kleinrock completed his: Queueing Systems in [1975]. Queueing systems can be modelled as network of queues, like waiting in line at a doctors clinic, railway reservation centre, in a bank, a bus stop, etc. A queueing system can be described as customers arriving for service, waiting for service, if the servers are busy and leaving the system after own service being finished. Gross and Harris [1985] studies Fundamentals of Queueing Theory. Little J.D.C. [1961] and W.S. Jewell [1967] developed the basic understanding of a queueing system and proof of the Little's formula. Little's theorem states as: "The average number of customers N can be determined by N = λt. Ke, J.C. and Wang K.H. (1999) analyzed cost analysis of the M / M / R machine repair problem with balking, reneging and server breakdown. Albin, S.L. (1984) analyzing M / M / 1 queues with perturbations in the arrival process. O.P. Sharma and A.M.K. Tarabia (2000) discussed a simple transient analysis of M / M / 1 / N queue.

Koo et al. (1995) discussed a manufacturing system modeling approach using computer spread sheet software where a static capacity planning model and stochastic queueing model are integrated. Several special manufacturing features like machine breakdown and batch production can be included in the model Wang, K.H. and Sivazlian B.D (1992) gave cost analysis of the *M* /*M* / R machine repair problem with spares operating under variable service rates. Zhang, H. and Shi, Dinghua discussed the M/M/1 queue with Bernoulli schedule. Jain, M. Rakhee and Maheshwari (2004) performed N-policy for a machine repair system with spares and reneging. M. Mursudi and H. Shafeek (2014) studied the application of queueing theory in multi-stage production line.

Where, λ = average arrival rate of units.
And, t = average service time for units.

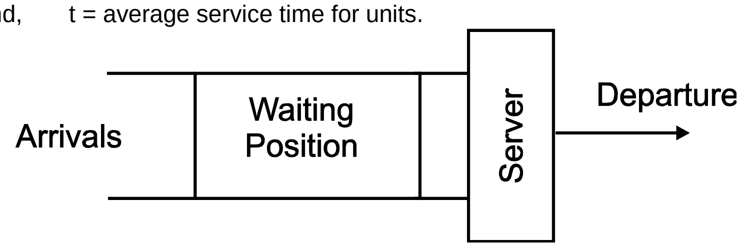

Figure: Standard graphical notation for queues.

In above figure, we have shown a standard graphical notation for queues. Until now, queueing systems have been well studied. For the shorthand for describing queueing processes, Kendall's [1953] notation was used.

M / M / 1 / N Queueing Model

In this queueing model arrival and service time distribution are Markovian, serie channel is 1 and capacity of this system is finite *N*.

Assumptions:

- λ is the arrival rate.
- μ is the service rate.
- P_n is the probability of n customers in the system.

For steady state, some performance measures:

1. Probability that there is no customer in the system is

$$P_0 = \begin{cases} \dfrac{(1-\lambda/\mu)}{1-(\lambda/\mu)^{N+1}}, & \text{if } \dfrac{\lambda}{\mu} \neq 1 \\ \dfrac{1}{1+N}, & \text{if } \dfrac{\lambda}{\mu} = 1 \end{cases} \quad (1)$$

2. Probability that there are n customers in the system

$$P_n = \begin{cases} \left(\dfrac{\lambda}{\mu}\right)^n P_0 & \text{if } n \le N \\ 0, & \text{if } n > N \end{cases} \quad (2)$$

3. Average Queue length or expected number of units in the system

$$E(n) = \sum_{n=0}^{N} nP_n = \begin{cases} \left(\dfrac{\lambda}{\mu}\right) \dfrac{\left\{1-(N+1)\left(\dfrac{\lambda}{\mu}\right)^N + N\left(\dfrac{\lambda}{\mu}\right)^{N+1}\right\}}{\left(1-\dfrac{\lambda}{\mu}\right)\left\{1-\left(\dfrac{\lambda}{\mu}\right)^{N+1}\right\}} & \text{if } \lambda \neq \mu \\ \dfrac{N}{2}, & \text{if } \lambda = \mu \end{cases}, \quad (3)$$

4. Average waiting length or expected number of units waiting for service

$$E(L) = \frac{\left\{1-N\left(\dfrac{\lambda}{\mu}\right)^{N-1} + (N-1)\left(\dfrac{\lambda}{\mu}\right)^N\right\}}{\left(1-\dfrac{\lambda}{\mu}\right)\left\{1-\left(\dfrac{\lambda}{\mu}\right)^{N+1}\right\}}\left(\frac{\lambda}{\mu}\right)^2 \quad (4)$$

5. The average waiting time in the queue

$$E(w) = \frac{E(L)}{\lambda'}$$

Where λ' is the new arrival rate. (5)

Matrix Geometric Method

In probability theory, the matrix geometric method is a method for the analysis of a quasi-birth-death process. Continuous time Markov chain whose transition rate matrices with a repetitive block structure. Harrison Peter G, Patel Naresh, M.[1992].

Method Description

This method required a transition rate matrix with tri-diagonal block structure which is given as follows:

$$Q = \begin{bmatrix} B_{00} & B_{01} & & & \\ B_{10} & A_1 & A_2 & & \\ & A_0 & A_1 & A_2 & \\ & & A_0 & A_1 & A_2 \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

Where, B_{00}, B_{01}, A_0, A_1, A_2 are matrices. To compute the stationary distribution π, we have $\pi Q = 0$ and the balance equations are considered for sub-vectors π_i

$$\pi_0 B_{00} + \pi_1 B_{10} = 0$$
$$\pi_0 B_{01} + \pi_1 A_1 + \pi_2 A_0 = 0$$
$$\vdots \quad \vdots \quad \quad \vdots$$
$$\pi_{i-1} A_2 + \pi_i A_1 + \pi_{i+1} A_0 = 0$$

We observe that

$$\pi_i = \pi_1 R^{i-1}$$

Above equation holds true where, R is the Neutes' rate matrix. Amussen, S.R. [2003] and Neutes, M [1981]. Numerically, we have

$$[\pi_0 \quad \pi_1]\begin{bmatrix} B_{00} & B_{01} \\ B_{10} & A_1 + RA_0 \end{bmatrix} = [0 \quad 0]$$

By using this relation, we can find π_0, π_1, and calculate iteratively all the π_i.

Some Applications of Queueing Theory

Queueing theory or waiting lines is useful in traffic control and planning, finding the sequence of computer operations, telecommunication like waiting calls when server is busy, air port traffic, capacity planning for buses and trains, health services (e.g. control of hospital bed assignments). We have application of queueing theory in machine repairing system.

Numerical Example

At a doctor's clinic patients arrive at an average rate of 6 per hour. The consultancy time per entered patient is exponentially distributed with an average of 10 minutes per patient. Clinic has fixed capacity of patients i.e. at any time 8 patients are waiting no new patient is allowed for waiting. At the steady state of this queue find the expected (average) number of waiting patient.

Solution: This model is M / M /1 / N model

N = 8 (capacity of the system)

Arrival rate λ = 6 per hour = $\dfrac{1}{10}$ per min.

(Consultancy time) service rate $\mu = \dfrac{1}{10}$ per patient

Since $\lambda = \mu$

Therefore, expected number of patients waiting = $\dfrac{8}{2} = 4$

Conclusion

In this paper we have explained the queueing theory and its applications in our living system. We have discussed finite capacity M / M / 1 / N queueing model, performance measures and matrixes geometric method. We have explained some applications of queues with numerical example.

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